REGULARIZING PROPERTIES OF NONLINEAR ITERATIVE METHODS AND THEIR USE IN SOME CONVERSE PROBLEMS

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UDC 517.988.68:536.2

The regularizing properties of nonlinear iterative methods for solution of type I equations with self-conjugate operator are studied.

Mathematical modeling of complex heat-exchange processes involving converse problems has recently attracted much attention [1, 2]. It is known that the most effective approach to solution of incorrect problems is the use of regularizing (in the sense described by Tikhnov [3]) algorithms. Together with Tikhonov's variation regularizing algorithm [3], nonlinear iterative methods are also used: accelerated release, conjugate gradients, etc. One of the major problems in the theory of iterative algorithms is establishment of their regularizing properties. We will note that the stability of nonlinear iteration processes was considered in [4-7]. The present study will present a new family of iterative regularizing algorithms and a numerical example of thermal flux determination from temperature measurements within a specimen.

The various converse thermal conductivity problems (see [2]) reduce to solution of the equation

$$Ax = f,\ f \in R(A) \subset H,\ A^* = A \geqslant 0, \tag{1}$$

where A:H → H is a linear finite operator; H is a real Gilbert space; R(A) is the range of values of the operator A, which is generally speaking, not limited, and the kernel of the operator A is nontrivial, $\ker A \neq \{0\}$. . Let $\hat{x} \perp \ker A$ be a solution of Eq. (1). Given exact input equations {A, f} for the approximate solution of Eq. (1) we may use the following iterative methods [5, 8]:

$$x_{k+1}^{(\beta)} = (\varepsilon_k^{(\beta)} E + A)^{-1} (\varepsilon_k^{(\beta)} x_k^{(\beta)} + f),\ x_0^{(\beta)} \in H,\ k = 0,\ 1,\ \ldots, \tag{2}$$

$$\varepsilon_k^{(\beta)} = a_k \frac{\langle A^{\beta+2} (Ax_k^{(\beta)} - f),\ Ax_k^{(\beta)} - f \rangle}{\langle A^{\beta+1} (Ax_k^{(\beta)} - f),\ Ax_k^{(\beta)} - f \rangle}. \tag{3}$$

Here $\langle \cdot, \cdot \rangle$ is a scalar product in H, $\{a_k\}$ is a specified numeric sequence, such that $0 < a_k \leqslant a_* < \infty$, $\beta \in [-1, \infty)$ is a fixed number, $Ex = x$, $x \in H$.

Statement 1. a) The sequence $\{x_k^{(\beta)}\}$ defined by Eqs. (2), (3) converges strongly (in the norm H) to the element $\hat{\hat{x}} = \hat{x} \oplus Px_0^{(\beta)}$, where P is the operator of projection of H on ker A and the inequality

$$\|x_{k+1}^{(\beta)} - x_0^{(\beta)}\| \leqslant \|\hat{\hat{x}} - x_0^{(\beta)}\| \leqslant \|\hat{x} - x_0^{(\beta)}\|,\ k = 0,\ 1,\ \ldots$$

is valid;

b) let $f \perp \ker A$, $a_k = 1$ for k = 0, 1, ... Then for any fixed $\beta \in [-1, \infty)$ the sequence $\varepsilon_k^{(\beta)}$, $k = 0, 1, \ldots$, decreases monotonically: $\varepsilon_{k+1}^{(\beta)} \leqslant \varepsilon_k^{(\beta)}$, $k = 0, 1, 2, \ldots$ Here $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ is the norm in the space H.

It follows from statement 1 that for any $\beta \in [-1, \infty)$ $\{R_k(x_0^{(\beta)})\}$(where $(R_k(x_0^{(\beta)})$ is the action operator on the right side of Eq.(1) of the f process of Eqs. (2), (3), having k steps $x_0^{(\beta)}$ is the zeroeth iteration) is an approximating sequence(in the terminology of [9]) for the exact set transformation A^{-1}, converse to A.

Let the right side of Eq. (1) be specified to an accuracy δ, i.e., we have f^δ from the sphere $\|f^\delta - f\| \leqslant \delta$. In order to demonstrate the regularizing properties of the family

Kemerovo State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 954-958, December, 1985. Original article submitted May 17, 1985.

002-0841/85/4906-1439$09.50

$\{R_k(x_0^{(\beta)})\}$, we must find the dependence $k = k(\delta)$ (see [3]), such that

$$\lim_{\substack{\delta\to 0\\ k(\delta)\to\infty}} \sup_{\|f^\delta - f\|\leqslant\delta} \|R_{k(\delta)}(x_0^{(\beta)})f^\delta - \hat{x}\| = 0.$$

In the case where $\{R_k(x_0)\}$ is an approximating family consisting of linear operators (for example $R_k(x_0)$ does not depend nonlinearly on the iterations $x_1, x_2, \ldots, x_k$ of any process), the regularizing algorithm can easily be constructed. Principal difficulties occur in constructing a regularizing algorithm when $R_k(x_0)$ are nonlinear operators (see [5]). Apparently there are two approaches which can be used in constructing regularizing algorithms on the basis of nonlinear iterative methods: the first is "quasilinearization" methods of the type of Eqs. (2), (3) using the function $\Psi(\delta)$ (see [5]), while the second is based on use of the iterative regularization method of [10]. We note that the second approach is quite universal and permits construction of regularization algorithms on the basis of nonlinear gradient methods: accelerated release, minimal discrepancy, and their analogs, as well as solution of nonlinear problems [10]. Here we will consider the second approach relative to the methods of Eqs. (2), (3).

Application of the iterative regularization principle to Eqs. (2), (3) (without disturbing generality, we assume that $\alpha_k \equiv 1$) leads to the expressions

$$\tilde{x}_{k+1}^{(\beta)} = (\tilde{\varepsilon}_k^{(\beta)}E + B_k)^{-1}(\tilde{\varepsilon}_k^{(\beta)}\tilde{x}_k^{(\beta)} + f^\delta),\ \tilde{x}_0^{(\beta)} \in H, \tag{4}$$

$$\tilde{\varepsilon}_k^{(\beta)} = \frac{\langle B_k^{\beta+2}(B_k\tilde{x}_k^{(\beta)} - f^\delta),\ B_k\tilde{x}_k^{(\beta)} - f^\delta\rangle}{\langle B_k^{\beta+1}(B_k\tilde{x}_k^{(\beta)} - f^\delta),\ B_k\tilde{x}_k^{(\beta)} - f^\delta\rangle}, \tag{5}$$

where $B_k = A + \alpha_k E$, $\alpha_k > 0$ is the iterative regularization parameter. It it then true that that

<u>Statement 2.</u> Let $\alpha_k > 0,\ \alpha_{k+1} < \alpha_k,\ k = 0, 1, \ldots;$

$$\lim_{k\to\infty}\alpha_k = 0,\quad \sum_{k=0}^{\infty}\alpha_k = \infty,\quad \lim_{k\to\infty}\frac{\alpha_k - \alpha_{k+1}}{\alpha_k\alpha_{k+1}} = 0,$$

and moreover, $\alpha_k = \alpha_k(\delta)$ is chosen such that $\lim\limits_{\substack{\delta\to 0\\ k\to\infty}} \delta/\alpha_k = 0.$. Then

$$\lim_{\substack{\delta\to 0\\ k\to\infty}} \sup_{\|f^\delta - f\|\leqslant\delta} \|\tilde{x}_{k+1}^{(\beta)} - \hat{x}\| = 0.$$

Thus, it has been shown that iteration processes Eqs. (4), (5) are regularizing with respect to Tikhonov's algorithms.

We note that the class of sequences $\{\alpha_k\}$ is quite broad: for example it contains sequences of the form $\alpha_k = c(1 + k)^{-1/2}$, $c > 0$ being a constant.

As a practical illustration of the efficiency of the methods under consideration we will examine the following converse thermal conductivity boundary problem (see [2]).

Let the temperature field $T(y, t)$ satisfy the differential equation

$$\frac{\partial T}{\partial t} = a^2\frac{\partial^2 T}{\partial y^2},\ 0 < y < \infty,\ 0 < t \leqslant t_1,$$

and the following conditions:

$$T(y, 0) = T_0,\ \lambda\frac{\partial T}{\partial y}(0, t) = -q(t).$$

It is assumed that T_0 is a constant quantity, and to an accuracy δ the temperature $T_1^\delta(t) = T(y_1, t)$:

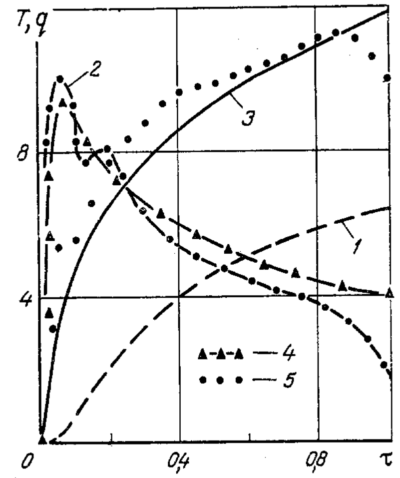

Fig. 1. Results of converse thermal conductivity problem solution for exact initial values (δ_0 = 0, curve 4) and for perturbed (δ_0 = 1%) but preliminarily smoothed cubic splines (curve 2): 1) $T_1^\delta(t)$; 2) tenth interation of method of [5]; 3) exact values of T(0, t); 5) calculated values of T(0, t).

$$\int_0^{t_1} [T_1^\delta(t) - T_1(t)]^2\, dt \leqslant \delta^2,\ T_1(t) = T(y_1,\ t),\ y_1 > 0.$$

It is required that we recreate the thermal flux density q(t). It is known that q(t) is defined by the following type I integral equation:

$$T_1^\delta(t) - T_0 = \frac{a\lambda}{\sqrt{\pi}} \int_0^t \exp\left[-\frac{y_1^2}{4a^2(t-\tau)}\right] \frac{q^{(\tau)}}{\sqrt{t-\tau}}\, d\tau. \qquad (6)$$

Perturbation of the initial $T_1(t)$ is accomplished in the following manner: $T_1^\delta(t) = T_1(t)(1+\delta_0\xi)$, where δ_0 is the value of the perturbation being modeled: $\xi \in [-1, 1]$ is a random value with equiprobable distribution law. To solve Eq. (6) we use methods of the type of Eqs. (2), (3), with the regularizing approximation chosen from discrepancy with *a priori* specification (depending on δ) of the number of iterations k = k(δ). The results of a = 1, t_1 = 1, x_0 = 0, y_1 = 0.25, δ_0 = 0 and δ_0 = 1%. In the example considered, use of unperturbed initial data gives a quite exact approximation to the original solution (see Fig. 1).

After the thermal flux density was reestablished the temperature $\tilde{T}(0, t)$ was calculated: at y_1 = 0 it was calculated by location of the integrals from Eq. (6). Results are presented in Fig. 1. We note that calculations were performed for other values of y_1: $0 \leqslant y_1 \leqslant 0{,}25$ and $\delta: 0 < \delta_0 \leqslant 5\%$, and in all cases results were obtained in 5-10 iterations. The numerical modeling procedure permits conclusions as to the efficiency of the methods considered for solution of converse problems.

NOTATION

A*, conjugate operator; k, iteration index; x_0, initial approximation; E, identity operator; δ, uncertainty level on right side of equation; T, temperature; t, time; a, thermal diffusivity coefficient; q, thermal flux density; λ, thermal conductivity coefficient; y, spatial coordinate; T_0, initial temperature distribution; t_1, maximum value of variable t; T_1^δ, perturbed temperature values; y_1, coordinate at which T_1^δ is measured; T(0, t), calculated temperature distribution.

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ALGORITHMS FOR ESTIMATING OPTIMUM DIMENSIONALITY OF AN APPROXIMATE SOLUTION OF THE CONVERSE THERMAL CONDUCTIVITY PROBLEM

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UDC 536.24

Algorithms are presented for calculating the optimum dimensionality of an approximate solution, using various *a priori* data on the uncertainty to which the right side of the operator equation is specified.

Formulation of the Problem. Many converse thermal conductivity problems reduce to solution of a type I operator equation [1]

$$K\varphi = f, \tag{1}$$

where $\varphi(x)$, $f(y)$ are functions of the spaces Φ, F; K is a completely continuous operator the null space of which is empty. The right side of $f(y)$ is specified by measurements at a discrete set $\{y_i\}$ of values $\tilde{f}_i = f(y_i) + \xi_i$, $i = 1, 2, \ldots, n$, where ξ_i is the random uncertainty (measurement noise) at the point y_i. It is necessary that we construct a solution of integral equation (1) from the initial data, $\{K, \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n\}$. As is well known, such a problem is incorrectly formulated [2], and various stable methods are used for its solution.

In a number of methods, for the approximate solution of Eq. (1) the element $\varphi_N(x)$ of a finite dimensional space Φ_N of dimensionality N is used [3]. The base functions of such a space may be either eigenfunctions of the operator K, or a set of some functions with good approximation properties. With such a construction of the approximate solution, the dimensionality N plays the role of a unique regularization parameter and determines the accuracy of the solution constructed. Choice of "suitable" dimensionality depends on both the level of uncertainty in the measurements, and the differential properties of the unknown solution. With reduced dimensionality the solution $\varphi_N(x)$ will not contain the "fine structure" of the

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Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 958-962, December, 1985. Original article submitted May 17, 1985.

0022-0841/85/4906-1442$09.50